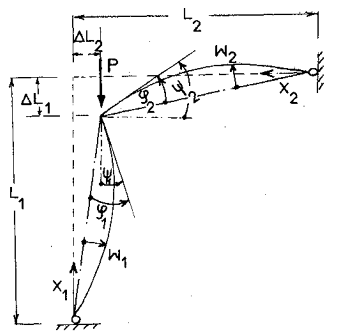

Fig. 1 Inextensional frame under axial load.

where

$$\varphi_l = w_l' \,|_{(x=\ell)} \tag{10}$$

and

$$\varphi_2 = \dot{w}_2 \,|_{(x=\ell)} \tag{11}$$

are the angles of rotation with $x_l = x_2$. Taking up to second-order terms, we obtain the auxiliary condition

$$a_2 = a_l - a_l^2 (\pi / 2\ell) \tag{12}$$

which, together with Eq. (6), forms an isoperimetric variational problem. Inserting Eqs. (1), (2), and (12) into Eq. (6), the isoperimetric problem can be reduced to a free variational problem given by

$$V = \frac{EI}{2}\left(a_l^2 \frac{\pi^4}{\ell^3} - \frac{P}{EI} a_l^2 \frac{\pi^2}{2\ell} - a_l^3 \frac{\pi^5}{2\ell^4} \right) \tag{13}$$

where the fourth-order terms have been neglected compared with the third-order terms. Differentiating Eq. (13) with respect to a_l and changing the perturbation parameter from a_l to φ_l (using $\varphi_l = \pi / \ell a_l$), the equation governing the initial post buckling behavior is obtained as

$$P = 19.7 (EI/\ell^2) - \tfrac{3}{4} (\pi/\ell)^2 EI \varphi_l \tag{14}$$

or‡

$$P/P^c = 1 - 0.375 \varphi_l \tag{15}$$

Thus, the post buckling behavior is unstable due to a non-vanishing initial post buckling slope. In other words, the system possesses an asymmetrical point of bifurcation and is, therefore, imperfection sensitive. The behavior resembles, in principle, the behavior of the cylindrical shell under axial pressure and the spherical shell under external pressure where the initial slope of the post buckling path was found to be nonzero (Fig. 2). It is important to note that the potential energy of the frame involves no stretching energy at all and that the nonvanishing third-order terms which caused the non-vanishing initial post buckling slope were introduced into the energy functional through the auxiliary nonlinear compatibility condition. This is in clear contrast to the elastic problems,[7] where the rise or fall of the post buckling path is due to the taking of the higher-order terms of the rotation into account.

Conclusion

The preceding analysis shows two important points. First, unstable post buckling behavior can arise due to an isoperimetric condition. This means that many of the recent publications concerning shell buckling require some modification, as they attribute unstable behavior solely to the role of the stretching energy.[10,15] Second, the buckling of equally extensional or inextensional frames does not, in general, occur under neutral equilibrium conditions.

‡Equation (15) is almost identical with that obtained by Koiter ($P/P^c = 1 - 0.3805\varphi_1$) using an exact and, therefore, more complicated analysis.

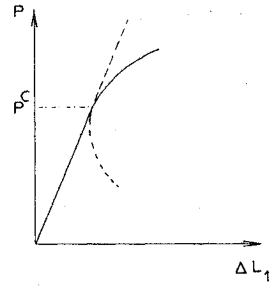

Fig. 2 Post critical behavior of the inextensional frame.

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Method of Integral Relations and Triple-Point Location in Impinging Jets

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Nomenclature

M_N = nozzle exit Mach number
p_a = ambient pressure

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p_N = nozzle exit plane pressure
q_{max} = maximum adiabatic velocity
q_{SB} = slip line velocity at its singular point
R_N = exit radius of nozzle
R_T = radius of triple point
y_{NP} = nozzle-to-plate separation distance
Δ_T = triple point height
μ_N = nozzle exit Mach angle

Introduction

THE problem of calculating the flowfield when a supersonic jet impinges on a perpendicular flat plate is a difficult one. Some success was achieved by Gummer and Hunt[1] in treating the impingement flow of uniform jets by a form of the Method of Integral Relations[2] (MIR). In the MIR, the continuity and axial momentum equations are integrated between the body and the shock and linear distributions are assumed for certain flow functions. A modified form of the continuity equation is normally used since the resulting approximate equations are found to be more accurate. Gummer and Hunt[3] also adapted the MIR to the case of underexpanded jets from conical nozzles. The results were only of limited value, the principal cause of difficulty being the slip line which is produced in the shock layer by the "triple point" intersection of the plate and jet shocks. Recognizing this problem, Belov, Ginzburg, and Shub[4] (referred to hereafter as BGS) have described a development of the MIR in which the outer part of the shock layer is treated by integrating between the plate and the slip line, for which a quadratic shape is assumed. The flow above the slip line (which has passed through the so-called "tail" shock) is not calculated. This Note reports the results of applying the BGS method to the apparently simpler case of an axisymmetric, overexpanded, initially uniform jet. Since the conclusions regarding the validity of the method are largely negative, only a brief outline of the treatment will be given here, together with the results of some experiments.

In the BGS method, the three coefficients for the quadratic form of the slip line are determined from the position and flow direction at the triple point and by satisfying a certain regularity condition which arises when the slip line becomes parallel to the plate. BGS ignored the triple point conditions which are given by the exact theory of three shock confluence points[5,6]; instead, they took the initial slip line conditions from the MIR calculation of the inner region, which is that region from the centerline to the triple point. This treatment ensures that the plate velocity will be continuous. In order to satisfy the regularity condition on the slip line, it is necessary to assume a value for the slip line velocity, q_{SB}, at the singular point. BGS take it to be sonic, presumably on the grounds that the slip line appears to form a throat at this point. (It is, however, easy to show that the area minimum occurs inboard of this point.) The satisfaction of the regularity condition also requires that the mass flow at the singular point be known: this means that the MIR must be based on the regular form of the equation of continuity, rather than on the more desirable modified form. The initial shock height is determined in the same manner as in the MIR treatment of a smooth blunt body, that is by applying a further regularity condition at the body sonic point. This has to be done by iteration. BGS do not offer a comparison with experimental measurements.

For the purposes of the present study, some schlieren pictures were taken on a rig which has been described in the thesis by Kalghatgi.[7] A number of plate pressure distributions were also available from another investigation.[8]

Results and Discussion

A study of the three shock intersection for normally impinging, overexpanded, initially uniform jets was reported earlier.[6] The work reported here was part of an attempt to calculate detailed shock shapes and surface pressures. This situation is in principle simpler than the underexpanded case for which the BGS method was originally formulated since the jet in the inner region is uniform.

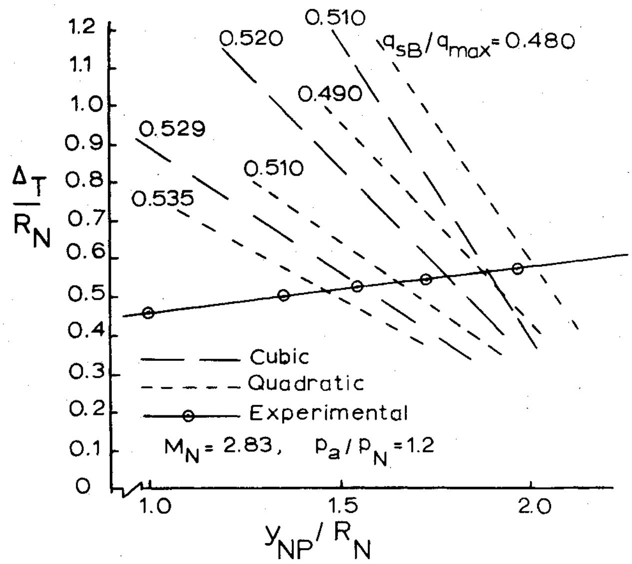

Fig. 1 Variation of triple point height with plate position.

In the first instance, the BGS method was applied in the form described in Ref. 4 and earlier in this section. However, no solutions could be obtained with this formulation; real coefficients for the slip line shape could only be obtained for values of shock height which were above the value at which the surface sonic regularity condition is satisfied. Following this, a number of variations were tried: the exact triple point conditions were used in place of the inner region MIR values, a low supersonic value was used for q_{SB} and the slip line shape was changed to a cubic (the extra condition applied was zero curvature at the point where it became parallel to the plate, since this appeared to be the case in schlieren pictures).

With the quadratic slip line, solutions could only be obtained if the exact triple point conditions and a supersonic value of q_{SB} were used. The results turned out to be very sensitive to the choice of the arbitrary quantity q_{SB}. Figure 1 shows predicted and experimental values of the height Δ_T of the triple point above the plate for a particular case. The sensitivity to the value of q_{SB} can be seen. It can also be seen that the predicted values for Δ_T decrease as the distance y_{NP} of the plate from the nozzle increases, which is opposite to the experimentally observed trend. The inner part of the plate pressure distributions is predicted reasonably well, which it must be since the centerline value and gradient are inevitably correct. However, the position of the sonic point on the plate turns out to be very sensitive to the value chosen for q_{SB}. For a fixed value of q_{SB}, the computed radial position of the sonic point decreases rapidly as y_{NP} increases, whereas the experimental results show it to be sensibly independent of the plate position. The predicted values are mostly substantially smaller than the experimental values.

Use of the cubic approximation for the slip line considerably extended the range of conditions under which solutions were obtainable. However, two of the major problems found with the quadratic approximation were still present; the solutions were very sensitive to the value of q_{SB} and the variations of the shock height and the radial location of the sonic point with y_{NP} were of the opposite nature to those observed experimentally. The variations of Δ_T can be seen in Fig. 1.

It is not possible to identify with certainty the reason for the disappointing performance to the BGS method in these cases. Part of the problem may lie in the use of the unmodified continuity equation. It seems more likely, however, that the basic defect is the neglect of the entire tail shock flow and with it the

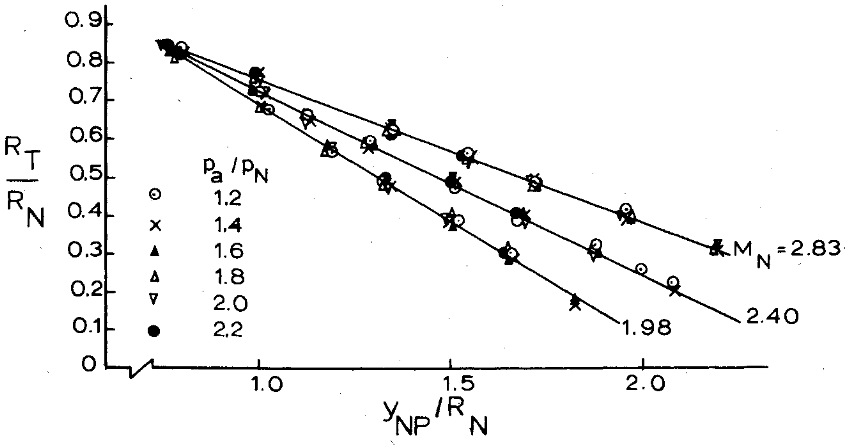

Fig. 2 Measured variation of triple point radius with plate position.

centered expansion fan which occurs at the edge of the jet shock. It is known that this centered expansion fan is the principal factor in determining the structure of the near wall jet.[9]

A Further Experimental Observation

As well as shock heights, the radial location, R_T, of the triple point was measured from the schlieren pictures. It was found that the value of R_T varied linearly with y_{NP} but was curiously insensitive to the overexpansion ratio, p_a/p_N. Figure 2 shows the results obtained with three different nozzle exit Mach numbers. These results do not mean that the shock structure is invariant with p_a/p_N. Indeed the jet shock angle increases as p_a/p_N increases and Δ_T must also increase in order to preserve a constant value of R_T. Even more surprisingly, in each case, the line through the experimental points of Fig. 2 makes an angle with the y_{NP} axis which equals the Mach angle, μ_N, in the exit plane. It can also be seen that each of the lines pass through the point $y_{NP}=0.75$, $R_T=0.845R_N$, thus enabling the entire set of results for R_T to be represented by the expression

$$R_T=(0.845+0.75\tan\mu_N)R_N-y_{NP}\tan\mu_N$$

From this, an expression for the triple point height Δ_T can be obtained. The jet shock angle changes slightly as it is propagated downstream but can be well represented by the arithmetic mean of its values at the nozzle lip (β_N) and at the triple point (β_T). Elementary geometry and the above equation then lead to the expression

$$\Delta_T=y_{NP}(1\text{-}\tan\mu_N\cot\bar{\beta})+R_N(0.75\tan\mu_N-0.155)\cot\bar{\beta}$$

where $\bar{\beta}\equiv 0.5(\beta_N+\beta_T)$. In many cases, it will be sufficient to take $\bar{\beta}=\beta_N$.

No physical explanation is offered for this curious collapsing of the data. Indeed, it may have no physical significance. However, it is a little easier to see what is happening if one observes that, if Δ_T were independent of y_{NP}, then the lines for R_T would make an angle with the y_{NP} axis equal to the shock angle: the fact that the angle μ_N occurs instead is due to the increase of Δ_T with y_{NP} which can be seen in Fig. 1.

Conclusions

The BGS method is not satisfactory for impinging overexpanded jets. The most likely reason for this being the neglect of the tail shock flow. Simple empirical expressions have been obtained for radial and axial locations of the triple point formed by the intersection of the plate shock with the jet shock.

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Attitude Control of Spinning Spacecraft by Radiation Pressure

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Nomenclature

A_i, ϵ_i = control plate area and moment arm, respectively, $i=1,2$

I_s, I_t = moments of inertia of the satellite about the symmetry and transverse axes, respectively

O = center of the Earth

S = center of mass of the satellite

$\bar{i},\bar{j},\bar{k}$ = unit vectors along x,y, and z axes, respectively

$\bar{n}_i$ = unit vector along plate normal, $i=1,2$

p = solar radiation pressure, 4.65×10^{-6} N/m^2

$\bar{u}$ = unit vector in the direction of the sun, $u_x\bar{i}+u_y\bar{j}+u_z\bar{k}$

u_x = $\cos\sigma\cos\phi+\sin\sigma\cos i\sin\phi$

u_y = $-(\cos\sigma\sin\phi-\sin\sigma\cos i\cos\phi)\cos\theta-\sin\sigma\sin i\sin\theta$

u_z = $(\cos\sigma\sin\phi-\sin\sigma\cos i\cos\phi)\sin\theta-\sin\sigma\sin i\cos\theta$

ρ = reflectivity of control surface

$\bar{\omega}$ = angular velocity of satellite, $\omega_x\bar{i}+\omega_y\bar{j}+\omega_z\bar{k}$

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